Solving k-Set Agreement Using Failure Detectors in Unknown Dynamic Networks

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Abstract—The failure detector abstraction has been used to solve agreement problems in asynchronous systems prone to crash failures, but so far it has mostly been used in static and complete networks. This paper aims to adapt existing failure detectors in order to solve agreement problems in unknown, dynamic systems. We are specifically interested in the k-set agreement problem.
The problem of k-set agreement is a generalization of consensus where processes can decide up to k different values. Although some solutions to this problem have been proposed in dynamic networks, they rely on communication synchrony or make strong assumptions on the number of process failures.
In this paper we consider unknown dynamic systems modeled using the formalism of Time-Varying Graphs, and extend the definition of the existing $\Pi\Sigma_{x,y}$ failure detector to obtain the $\Pi\Sigma_{\perp,x,y}$ failure detector, which is sufficient to solve k-set agreement in our model. We then provide an implementation of this new failure detector using connectivity and message pattern assumptions. Finally, we present an algorithm using $\Pi\Sigma_{\perp,x,y}$ to solve k-set agreement.

Index Terms—Distributed systems, Dynamic networks, Failure detectors, k-Set agreement

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1 INTRODUCTION

Dynamic distributed systems such as wireless or peer-to-peer networks pose new challenges to the field of distributed computing. In these systems, processes can join or leave the system during the run, and the communication graph evolves over time.

In unknown networks, processes are lacking initial information on system membership. Dynamic networks are often unknown, since it is difficult to know ahead of time which processes may join the system in the future.

Most of the existing distributed algorithms in the literature were meant for static, known networks and make assumptions that are unrealistic in the context of unknown dynamic networks: communication graphs are often assumed to be fully connected or even complete, and processes are expected to have full knowledge of the system membership. As a result, adapting existing protocols to unknown and/or dynamic networks is not trivial.

Agreement problems, and notably consensus, have been a lot less studied in dynamic networks than in static networks. In this paper we are interested in the k-set agreement problem, which is a generalization of the consensus problem such that 1-set agreement is consensus. In the k-set agreement problem, each process proposes a value, and some processes eventually decide a value while respecting the properties of validity (a decided value is a proposed value), termination (every correct process eventually decides a value) and agreement (at most k values are decided).

Protocols solving consensus or k-set agreement have been proposed for dynamic systems, but they assume synchronous communications (as in [1], [2], [3], [4]) or make strong assumptions on the number of process failures [5].

We approach the k-set agreement problem from a failure detector perspective [6], [7]. Failure detectors provide processes with information on process failures. They have been used as an abstraction of system assumptions to circumvent the impossibility of solving consensus in asynchronous systems prone to crash failures [8].

The $\Pi\Sigma_{x,y}$ failure detector was introduced in [9] and is sufficient to solve k-set agreement in static networks (if and only if $k \geq xy$) while being weaker than other known failure detectors which solve the same problem. However, this failure detector relies on information that is not available in unknown networks: the list of all the participating processes. Additionally, traditional failure detectors rely on a full connectivity of the network graph, which is not available in a dynamic network.

In the current paper we extend the definition of $\Pi\Sigma_{x,y}$ in order to obtain the $\Pi\Sigma_{\perp,x,y}$ failure detector, which is capable of solving k-set agreement in unknown dynamic systems, and provide implementations of this new detector. We also adapt the k-set agreement algorithm of [9], [10] to solve k-set agreement using $\Pi\Sigma_{\perp,x,y}$ on top of our model.

The model assumptions we propose to implement $\Pi\Sigma_{\perp,x,y}$ are generic and expressed in terms of message pattern, which allows our model to be applied to a range of systems. We also provide concrete examples of partial synchrony and failure pattern properties which are sufficient to ensure our generic assumptions.

The system is modeled using the formalism of the Time-Varying Graph (TVG), as defined in [11].

This paper thus brings the following main contributions:
1) The definition of the $\Pi\Sigma_{\perp,x,y}$ failure detector as an adaptation of $\Pi\Sigma_{x,y}$ to solve k-set agreement in unknown dynamic networks.
2) An algorithm implementing $\Pi\Sigma_{\perp,x,y}$ in our model, with connectivity and message pattern assumptions.

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3) An algorithm solving k-set agreement in our model enriched with $\Pi\Sigma_{\perp,x,y}$.

The remaining of the paper is organized as follows: Section 2 formally describes our system model. Section 3 presents the definitions of several failure detectors relevant to our work, and introduces the $\Pi\Sigma_{\perp,x,y}$ failure detector. Section 4 defines the different connectivity and message pattern assumptions that we rely on. In Section 5, we propose an implementation of $\Pi\Sigma_{\perp,x,y}$. Section 6 presents an algorithm solving k-set agreement with $\Pi\Sigma_{\perp,x,y}$ for $k \geq xy$. Section 7 presents the related work. Finally, Section 8 concludes the paper.

2 SYSTEM MODEL

2.1 Process Model

A finite set of n processes $\Pi = \{p_1, ...p_n\}$ participate in the system. *The processes are synchronous* (there is a bound on the relative speed of processes) and uniquely identified, although initially they are only aware of their own identities. Processes are not required to know the value of n.

A run is a sequence of steps executed by the processes while respecting the causality of operations (each received message has been previously sent). Processes can join and leave the system during the run (Π is the set of all processes that participate in the system at some point in time). Processes may also crash, and we make no difference between a process that crashes permanently and a process that leaves the system permanently: in both cases the process is considered *faulty* in that run. A process that is not faulty is called *correct*. Note that this definition of faulty and correct processes is not exactly the traditional one. Indeed, correct processes can crash or leave the system, as long as they recover or come back later. Only processes that crash or leave permanently are considered faulty.

Correct processes can leave the system and come back infinitely often, but they can only crash and recover a finite number of times. The critical difference is that a process that leaves the system keeps its memory intact, whereas a crashed process does not.

The set of all correct processes is called $\mathcal{C}$. We assume a bound $f < n$ on the number of faulty processes in a run.

2.2 Communication Model

Processes communicate by sending and receiving messages. *Communications are asynchronous*: there is no bound on message transfer delays. Therefore, even though processes are synchronous, they do not cooperate in a synchronous way.

The system is dynamic, which means that nodes and communication links can appear or disappear during the run: therefore, the communication graph will change over time. The usual notion of path in the graph is not sufficient to define reachability in such a dynamic graph. To solve this issue, several solutions were proposed in the literature [1], [11], [12], [13]. Among them, we choose to model the communication graph using the Time-Varying Graph (TVG) formalism, as defined by Casteigts et al. in [11].

2.2.1 Time-Varying Graphs

Definition 1 (Time-Varying Graph). *A time-varying graph is a tuple $\mathcal{G} = (V, E, \mathcal{T}, \rho, \zeta, \psi)$ where*

1) $V = \Pi$ is the set of nodes in the system.
2) $E \subseteq V \times V$ is the set of edges.
3) $\mathcal{T} = \mathbb{N}$ is a time span.
4) $\rho : E \times \mathcal{T} \rightarrow \{0, 1\}$ is the edge presence function, indicating whether a given edge $e \in E$ is active at a given time $t \in \mathcal{T}$.
5) $\zeta : E \times \mathcal{T} \rightarrow \mathbb{N}$ is the latency function, indicating the time taken to cross an edge $e \in E$ if starting at given time $t \in \mathcal{T}$.
6) $\psi : V \times \mathcal{T} \rightarrow \{0, 1\}$ is the node presence function, indicating whether a given node $p \in V$ is present in the system at a given time $t \in \mathcal{T}$. The edge presence function and the node presence function must be coherent: $\forall t \in \mathcal{T}, \forall p_i \in V, \forall e \in E$, if e is connected to p_i then $\psi(p_i, t) = 0 \implies \rho(e, t) = 0$.

$G(V, E)$ is the underlying graph of $\mathcal{G}$, and indicates which nodes have a relation at some time in $\mathcal{T}$.

Note that processes do not know the values of the ζ function, which is only introduced for the simplicity of presentation. Since communications are asynchronous, the values of ζ are finite but not necessarily bounded.

The communication links between processes are not permanent: the ρ function indicates when a given edge is active. Therefore, the usual notion of *path* in the graph is not suited to TVGs: journeys are defined for this purpose.

Intuitively, a journey is a path over time. In order to transmit a message from process p_i to process p_j, it is not necessary for every edge on the path to be active at the time the message is sent: it is sufficient that there exists a path between between p_i and p_j such that all the edges on the path are active in the right order at some time in the future.

Definition 2 (Journey). *A journey is a sequence of couples $\mathcal{J} = \{(e_1, t_1), (e_2, t_2), ..., (e_m, t_m)\}$ such that $\{e_1, e_2, ..., e_m\}$ is a walk in G and:*

$$\forall i, 1 \leq i < m : (\rho(e_i, t_i) = 1) \wedge (t_{i+1} \geq t_i + \zeta(e_i, t_i)) \ .$$

t_1 is called $departure(\mathcal{J})$ and $t_m + \zeta(e_m, t_m)$ is called $arrival(\mathcal{J})$. We denote $\mathcal{J}^*_{(u,v)}$ the set of all the journeys starting at node u and ending at node v.

Consider the following example: a graph where $E = V \times V$ and every edge in the system is active infinitely often (longer than the message transfer time), but no more than one edge is ever active at a time. In such a system, there are journeys infinitely often between every node and the connectivity is sufficient to solve complex problems such as consensus. However, at any given instant, the graph is partitioned into at least $n - 1$ independent subsets. This shows that similarly to paths, the usual notion of graph partitioning loses relevancy in TVGs, since the number of partitions at a particular instant in the run is not a very useful parameter. Instead, we are interested in the number of *partitions over time*. In the rest of the paper, we use the word *partition* to refer to a subset of the network that is isolated from the rest of the network for an arbitrarily long duration, and not just temporarily.

2.2.2 Communication primitive

Processes communicate exclusively by sending messages with a very simple broadcast primitive. When a process p_i calls the broadcast primitive, the message is simply sent to the processes that are currently in p_i's neighborhood, including p_i. The broadcast is not required to provide advanced features such as message forwarding, routing, message ordering or any guarantee of delivery.

2.2.3 Channels

The channels are fair-lossy. Messages may be lost but, if the edge is active for the entire time of the message transfer, a message sent infinitely often will be received infinitely often. Messages may be duplicated, but a message may only be duplicated a finite number of times. No message can be created or altered. We make no assumption on message ordering and do not require channels to be FIFO.

3 FAILURE DETECTORS

Failure detectors ([6]) are distributed oracles that provide processes with information on process failures, often in the form of a list of trusted process identities. This information is unreliable in the sense that the failure detector may erroneously consider a correct process as faulty, or vice versa, but will attempt to correct these mistakes later. Each failure detector class ensures some properties on the reliability of the failure information. A failure detector is an abstraction of the system assumptions used to solve a given problem.

The failure detector abstraction has been investigated as a way to circumvent the impossibility result of [8] and solve consensus in asynchronous systems prone to crash failures [7]. Our goal in this paper is to adapt this solution to solve k-set agreement in dynamic systems.

Traditionally, failure detectors are used in system models considering static and fully connected communication graphs. These connectivity properties are usually presented as properties of the system model rather than the failure detector augmenting it. When considering a much weaker system model such as a dynamic network, solving any non-trivial problem still requires the assumption of a certain degree of graph connectivity, as not much can be done in a system where no communication link is ever active. Studying dynamic systems means considering the level of temporal connectivity required to solve a specific problem, and using a generic and strong connectivity assumption would defeat that purpose. Instead, the goal should be to use a weak connectivity assumption that is still sufficient to solve the problem. Therefore, to solve a given agreement problem, two things are necessary: (1) a failure detector and (2) connectivity assumptions.

But if connectivity assumptions must be added to the system model in addition to the failure detector, then it cannot be said that the failure detector is *sufficient* to solve the problem. For this reason, and because in a dynamic system the required level of connectivity is as dependent on the problem as the required failure detector, we consider that failure detectors for dynamic systems should include connectivity properties. Adding these connectivity properties should not be seen as strengthening the failure detectors, they are still weaker than the assumption of a fully connected, static communication graph.

Additionally, our system model considers an unknown network where processes have no information on system membership at the beginning of the run. A way to circumvent this issue was proposed in [14] in the form of the $\Sigma_\perp$ failure detector

Our approach is based on the $\Pi\Sigma_{x,y}$ failure detector of [9], augmented with connectivity properties and extended with the method of [14] in order to obtain a failure detector sufficient to solve the k-set agreement problem in unknown dynamic systems.

3.1 The Quorum Failure Detectors

The quorum failure detector Σ, [15], provides every process with sets of process identities (called quorums) such that any two quorums output by Σ at any time necessarily intersect. Additionally, Σ requires that all quorums eventually contain only correct processes.

The Σ_k failure detector, [16], is a generalization of Σ meant to solve k-set agreement. Similarly to Σ, Σ_k provides processes with eventually correct quorums, and at least two out of any $k+1$ quorums intersect. It follows that $\Sigma = \Sigma_1$.

Intuitively, Σ_k prevents the network from partitioning into more than k independent subsets. Note that in the case of a dynamic network, this statement only applies to *partitions over time*: the network may still be instantaneously partitioned into any number of subsets at any given instant.

In message passing systems, Σ is necessary for consensus ([15]) and Σ_k is necessary for k-set agreement ([16]).

The intersection property of both Σ and Σ_k must hold over time, which means that if a process queries its failure detector before any communication has taken place, the returned quorum must intersect with the quorums formed by processes later in the run. In known networks, implementations of Σ traditionally solve this issue by returning Π as a quorum at the beginning of the run [15]. This is not an option in unknown networks where system membership knowledge is only established through communication.

The $\Sigma_\perp$ failure detector ([14]) is an adaptation of Σ for unknown networks. Instead of returning a quorum, $\Sigma_\perp$ can also output the default value $\perp$ whenever the knowledge necessary to form a quorum has not been gathered yet.

In order to solve k-set agreement in unknown dynamic networks, we define the $\Sigma_{\perp,k}$ failure detector, which combines the properties of Σ_k and $\Sigma_\perp$. It also includes a connectivity property which replaces (and is weaker than) the assumption of a static and complete network.

The $\Sigma_{\perp,k}$ failure detector provides each process p_i with a quorum denoted qr_i^τ (which is either a set of process identities or the special value $\perp$) at any time instant τ.

For the convenience of the presentation, we introduce the following definition:

Definition 3 (Recurrent neighborhood)**.** *The recurrent neighborhood of a correct process p_i, denoted R_i, is the set of all correct processes whose quorums intersect infinitely often with p_i's quorums.* $\forall p_i \in \mathcal{C}, R_i = \{p_j \in \mathcal{C} \mid \forall\tau, \exists\tau_i, \tau_j \geq \tau : qr_i^{\tau_i} \neq \perp \wedge qr_j^{\tau_j} \neq \perp \wedge qr_i^{\tau_i} \cap qr_j^{\tau_j} \neq \emptyset\}$.

Note that $p_j \in R_i$ is an equivalence relation between p_i and p_j. By definition, $\forall p_i \in \mathcal{C} : p_i \in R_i$, therefore $R_i \neq \emptyset$.

We say that a correct process p_i can *reach* another correct process p_j if, provided that p_i sends messages infinitely often, p_j receives them infinitely often.

$\Sigma_{\perp,k}$ is defined by the self-inclusion, quorum liveness, quorum intersection and quorum connectivity properties.

Property 1 (Self-inclusion)**.** *Every process includes itself in its non-$\perp$ quorums.* $\forall p_i \in \Pi, \forall\tau : (qr_i^\tau \neq \perp) \implies (p_i \in qr_i^\tau)$.

Property 2 (Quorum liveness)**.** *Eventually, every correct process stops returning $\perp$ and its quorums only contain correct processes.* $\exists\tau, \forall p_i \in \mathcal{C}, \forall\tau' \geq \tau : qr_i^{\tau'} \neq \perp \wedge qr_i^{\tau'} \subseteq \mathcal{C}$.

Property 3 (Quorum intersection)**.** *Out of any $k+1$ non-$\perp$ quorums, at least two intersect.*

$$\forall \tau_1, ..., \tau_{k+1} \in \mathcal{T}, \forall id_1, ..., id_{k+1} \in \Pi,$$
$$\exists i, j : 1 \leq i \neq j \leq k+1 :$$
$$(qr_{id_i}^{\tau_i} \neq \perp \wedge qr_{id_j}^{\tau_j} \neq \perp) \implies (qr_{id_i}^{\tau_i} \cap qr_{id_j}^{\tau_j} \neq \emptyset) \ .$$

Property 4 (Quorum connectivity)**.** *Every correct process p_i can* reach *every process in R_i.*

Σ_k and $\Sigma_\perp$ were defined with only 2 properties (liveness and intersection). Self-inclusion is a property added to Σ_x and $\Pi\Sigma_x$ by the authors in [9] for the sake of the simplicity of algorithm proofs, and it is trivially implemented by the algorithms we present in the paper. Quorum connectivity is the property added to deal with network dynamicity.

Intuitively, the quorum connectivity property means that processes belong to the same partition as their recurrent neighborhood. Note that $\forall p_i, p_j \in \mathcal{C} : p_i \in R_j \implies p_j \in R_i$, thus quorum connectivity enables two-way communication between p_i and p_j. This property is not very costly, since most failure detector implementations already require some level of connectivity between processes in a quorum in order to form the quorums themselves. This is the case for the $\Sigma_{\perp,k}$ algorithm we present in Section 5, which does not require any additional assumption to implement quorum connectivity.

3.2 The Family of Failure Detectors $\Pi\Sigma_{x,y}$

Although Σ_k is necessary to solve k-set agreement, it is not sufficient. It has been shown in [10] that k-set agreement can be solved in static asynchronous networks with $\langle \Sigma_x, \overline{\Omega}_y \rangle$, with $k \geq xy$, where $\overline{\Omega}_y$ is the eventual anti-leader detector [17]. It was shown in the same paper that if $n \geq 2xy$, then there is no $\langle \Sigma_x, \overline{\Omega}_y \rangle$-based k-set algorithm for $k < xy$, which means that the $k \geq xy$ requirement is tight.[1]

However in [9], Mostéfaoui, Raynal and Stainer introduce the $\Pi\Sigma_{x,y}$ failure detector and prove that it is strictly weaker than $\langle \Sigma_x, \overline{\Omega}_y \rangle$ for $1 < y < x < n$ while still being strong enough to solve k-set agreement with $k \geq xy$. Interestingly, $\Pi\Sigma_{x,y}$ is defined incrementally based on the properties of Σ_x. Therefore, an algorithm for Σ_x (or $\Sigma_{\perp,x}$, in our case) can easily be extended to implement $\Pi\Sigma_{x,y}$ (resp., $\Pi\Sigma_{\perp,x,y}$), with an additional assumption.

The authors in [9] provide an intuitive description of $\Pi\Sigma_{x,y}$. $\Pi\Sigma_{x,1}$ (1) prevents the system from partitioning into more than x partitions with the properties of Σ_x and (2) guarantees that the processes of at least one of these subsets agree on a common leader. $\Pi\Sigma_{x,y}$ can be seen as y independent instances of $\Pi\Sigma_x$ in which (2) has to be guaranteed in only one of these instances.

We define $\Pi\Sigma_{\perp,x,y}$ as an extension of $\Pi\Sigma_{x,y}$ that includes the properties of $\Sigma_{\perp,x}$ and is capable of solving k-set agreement in unknown dynamic systems.

3.3 The Family of Failure Detectors $\Pi\Sigma_{\perp,x,y}$

Similarly to [9], $\Pi\Sigma_{\perp,x,y}$ is defined incrementally: $\Pi\Sigma_{\perp,x}$ is defined firstly.

3.3.1 The failure detector $\Pi\Sigma_{\perp,x}$

At any time instant τ, $\Pi\Sigma_{\perp,x}$ provides each process p_i with a quorum denoted qr_i^τ (which is either a set of process identities or the special value $\perp$) and a leader denoted $leader_i^\tau$ (which is a process identity).

$\Pi\Sigma_{\perp,x}$ is defined by the following properties:

- Self-inclusion
- Quorum liveness
- Quorum intersection
- Quorum connectivity
- Eventual partial leadership

$\Big\} \ \Sigma_{\perp,x}$

First, we define an eventual partial leader as follows:

Definition 4 (Eventual partial leader)**.** *An eventual partial leader p_l is a correct process such that every process in the recurrent neighborhood of p_l eventually recognizes p_l as its leader forever. $p_l \in \mathcal{C} \wedge \forall p_i \in R_l, \exists \tau, \forall \tau' \geq \tau : leader_i^{\tau'} = p_l$.*

We denote L the set of all eventual partial leaders.

Property 5 (Eventual partial leadership)**.** *For every correct process p_i, there is an eventual partial leader p_l that can* reach *p_i.*

The original eventual partial leadership property used in [9] simply requires the existence of an eventual partial leader in the system. Our version of the property similarly implies that $L \neq \emptyset$ (since $\mathcal{C} \neq \emptyset$), but also implies that each correct process must be reachable by one eventual partial leader (which, depending on the level of connectivity, may require more than one leader). In a static and connected network, both properties are equivalent: a single eventual partial leader is necessary and sufficient to fulfill the property, since the connected communication graph enables this single leader to reach every correct process.

In a k-set agreement algorithm, the eventual partial leaders are those processes that eventually decide. In order to ensure termination, the deciding leaders must, therefore, be able to inform the rest of the system of their decision. However, in a dynamic network, the mere existence of an eventual partial leader does not provide the latter with the necessary connectivity to guarantee termination. This is why in dynamic networks, our eventual partial leadership property is stronger than the original one and imposes the required connectivity.

The eventual partial leadership property implies a trade-off between the number of eventual partial leaders in the system and graph connectivity. On the one hand, if there is a single leader in the system, then this leader must be able to reach every correct process in the system. On the other hand, if the communication graph is partitioned, then there must be at least one local leader per partition.

Such a trade-off implies that the eventual partial leadership property does not prevent the system from being partitioned into up to n partitions over time, provided that every correct process identifies itself as its own eventual partial leader. However in this scenario it would be impossible to verify the quorum intersection and quorum connectivity properties.

3.3.2 The failure detector $\Pi\Sigma_{\perp,x,y}$

The definition of $\Pi\Sigma_{\perp,x,y}$ is the same as $\Pi\Sigma_{x,y}$ in [9], except that it uses $\Pi\Sigma_{\perp,x}$ instead of $\Pi\Sigma_x$. $\Pi\Sigma_{\perp,x,y}$ can be seen as y

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[1] This result can also be proved using the impossibility of k-set agreement theorem ([18]), the premise of which applies if $k < xy$.

instances of $\Pi\Sigma_{\perp,x}$ running concurrently.

$\Pi\Sigma_{\perp,x,y}$ provides each process p_i with an array $FD_i[1..y]$ such that for each j, $1 \leq j \leq y$, $FD_i[j]$ is a pair containing a quorum $FD_i[j].qr$ and a process index $FD_i[j].leader$. The array satisfies the following properties:

Property 6 (Vector safety)**.** $\forall j \in [1..y]$: *$FD_i[j].qr$ satisfies the self-inclusion, liveness, intersection and quorum connectivity properties of $\Pi\Sigma_{\perp,x}$.*

Property 7 (Vector liveness)**.** $\exists j \in [1..y]$: *$FD_i[j]$ satisfies the eventual partial leadership property of $\Pi\Sigma_{\perp,x}$.*

The idea is to reduce the cost of the system assumptions: the liveness property only needs to be verified by one out of a set of y instances of the detector.

The authors in [9] prove that for $1 \leq y \leq n-1$, $\Pi\Sigma_{1,y}$ is as strong as $\langle \Sigma_1, \overline{\Omega}_y \rangle$. This shows that the y parameter of $\Pi\Sigma_{x,y}$ (and $\Pi\Sigma_{\perp,x,y}$) is comparable to the y of $\overline{\Omega}_y$.

$\Pi\Sigma_{\perp,x,y}$ is sufficient to solve the k-set agreement problem in our model if $k \geq xy$. We will prove this by providing a k-set agreement algorithm relying on $\Pi\Sigma_{\perp,x,y}$ in Section 6.

4 ASSUMPTIONS

In this section we present some system assumptions. The algorithms presented in Section 5 will then list the assumptions from this section on which they rely.

4.1 Time-Varying Graph Classes

In addition to defining the formalism of the TVG, Casteigts et al. present in [11] a number of TVG classes which provide different levels of connectivity assumptions. We are particularly interested in class 5.

Definition 5 (Class 5: recurrent connectivity [11])**.** *All processes can reach each other infinitely often through journeys.* $\forall u, v \in \Pi, \forall \tau, \exists \mathcal{J} \in \mathcal{J}^*_{(u,v)} : departure(\mathcal{J}) > \tau$.

This connectivity assumption does not exactly fit the requirements of the proposed algorithms. On the one hand, it is too strong. It implies a global connectivity between any two processes in the system, which is not necessary to solve k-set agreement, since the problem can be solved in a system partitioned into k subsets. On the other hand, class 5 is too weak since it relies on the notion of journey, which is insufficient to ensure the transmission of messages. Even if a journey exists between p_i and p_j, there is no guarantee that a message sent by p_i can reach p_j. In fact, even if the edge between p_i and p_j is active infinitely often and the message is sent infinitely often, the message might always be sent in between two activation periods of the edge, thus never crossing it. To solve this problem, Gómez-Calzado et al. defined in [19] the notion of timely journeys for the case of synchronous systems. We extend this solution into γ-journeys for the case of asynchronous communications.

Definition 6 (γ-Journey)**.** *A γ-journey $\mathcal{J}$ (where $\gamma > 0$ is a time duration) is a journey such that every node on the path can wait up to γ units of time after the next edge becomes active before forwarding the message. Since the message may be sent at any time within the γ time window and the channel latency may vary during that time, the edge must remain active long enough for the worst case duration.*
- $\forall i, 1 \leq i \leq |\mathcal{J}|$, *$e_i$ stays active from time t_i until, at least, time $t_i + max_{0 \leq j \leq \gamma}\{j + \zeta(e_i, t_i + j)\}$.*
- $\forall i, 1 \leq i < |\mathcal{J}|, t_{i+1} \geq t_i + max_{0 \leq j \leq \gamma}\{j + \zeta(e_i, t_i + j)\}$.

With a γ-journey, processes are given an additional time window of γ units of time to send the message. In [19], this time was used to detect the activation of the edge. This solution is appropriate for point-to-point communications in a known network, since it allows the sender of the message to resend the message to the receiver whenever the edge appears again. However, this is not helpful in an unknown non-complete network where processes have to rely on blind broadcasts and forwarding to propagate information.

Instead, we use the time window provided by γ-journeys as an upper bound on the time between two transmissions of the message. This explains the need for synchronous processes: each process should be able to repeatedly send every message at least once every γ units of time.

Provided that processes receive their own broadcasts within γ units of time and then rebroadcast it, it is ensured that every message is sent at least once every γ units of time. If there is infinitely often a γ-journey from p_i to p_j, then p_i can reach p_j.

We call $\mathcal{J}^\gamma_{(u,v)}$ the set of all the γ-journeys from u to v.

Using class 5 as a starting point, we define TVGs of class 5-(α, γ) as follows. γ is the time duration parameter of γ-journeys, and α is a parameter defining the number of correct processes that each correct process is ensured to communicate with.

Assumption 1 (Class 5-(α, γ): (α, γ)-recurrent connectivity)**.** *Every correct process can reach and be reached through γ-journeys infinitely often by at least α correct processes.*

$$\begin{aligned} &\forall p_i \in \mathcal{C}, \exists P_i \subseteq \mathcal{C}, |P_i| \geq \alpha, \forall t \in \mathcal{T}, \forall p_j \in P_i, \\ &\quad \exists \mathcal{J}_i \in \mathcal{J}^\gamma_{(p_i,p_j)} : departure(\mathcal{J}_i) \geq t \ \wedge \\ &\quad \exists \mathcal{J}_j \in \mathcal{J}^\gamma_{(p_j,p_i)} : departure(\mathcal{J}_j) \geq t \ . \end{aligned}$$

This assumption is parametrized by the two values α and γ. A low γ value weakens the connectivity assumption by allowing shorter time windows for the journeys, but implies that processes must be able to send messages more often to ensure that a message is sent within the shorter window. On the other hand, a high γ value reduces the number of journeys that are qualified as γ-journeys, thus strengthening the connectivity assumption, but accepts slower processes.

The α parameter also presents a trade-off: class 5-(α, γ) indirectly implies that there must be at least α correct processes in the system. As a result, a high α value will result in a strong assumption on the number of process failures which can be costly in a dynamic system. A low α value would strengthen the message pattern assumptions presented in the next section.

Class 5-(α, γ) also implies that all correct processes must know a lower bound for α.

To summarize, the assumption of a TVG of class 5-(α, γ) means that correct processes are able to communicate infinitely often with a subset of α correct processes. This property ensures that correct processes will not wait for messages forever, which enables our algorithm to ensure the quorum liveness property. Additionally, if the algorithm ensures that every correct process p_i eventually only forms quorum from the P_i set, then class 5-(α, γ) also ensures quorum connectivity.

4.2 Message Pattern Assumptions

In this section we present message pattern assumptions, as defined by Mostéfaoui et al. in [20]. The message pattern model consists in assuming some properties on the relative order of message deliveries. If processes periodically wait for a certain number of messages, the idea is to assume that the message sent by some specific process will periodically be among the first ones to be received.

In order to express our message pattern assumptions, we assume that the distributed algorithm executed by processes uses a query-response mechanism. Processes periodically issue query messages, to which other processes respond.

The principle of our failure detector algorithm revolves around processes repeatedly issuing a query and then waiting for responses from α processes. The α parameter is therefore the minimum size of quorums returned by the algorithm, which does not necessarily constitute an assumption on the number of failures, since α might be equal to 1. Note that α is the same parameter we used to define TVGs of class 5-(α, γ) which ensures that correct processes will not wait for messages infinitely.

We call *response set* the first α processes whose response to a given query from process p_i are received by p_i.

4.2.1 Assumption for Quorum Intersection

The assumption of a TVG of class 5-(α, γ) is not sufficient to ensure the quorum intersection property. In [10], Bouzid and Travers proposed a method to implement quorums: if processes repeatedly wait for messages from at least $\lfloor \frac{n}{k+1} \rfloor + 1$ processes before outputting these processes as their new quorum, then the size of quorums alone is sufficient to ensure intersection. This method implies that there must be at least $\lfloor \frac{n}{k+1} \rfloor + 1$ correct processes in the system, otherwise processes would wait forever, thus preventing liveness.

In a dynamic system where processes are expected to join and leave the system, an assumption on the number of process failures seems too costly. For this reason, our failure detector algorithms rely on the message pattern approach.

The following assumption is sufficient for our algorithm to implement quorum intersection. It was obtained by generalizing the assumption used for the case $k = 1$ in [14].

Assumption 2 (Generalized winning quorums)**.** *$\exists m \in [1, k]$ and $\exists Q_{w1}, ..., Q_{wm} \subseteq \Pi$ (called winning quorums). Each winning quorum Q_{wi} is associated with a number $w_i \geq 1$ (called the* weight *of Q_{wi}) such that $\sum_{i=1}^{m} w_i \leq k$. $\forall p \in \Pi$, every time p issues a new query, $\exists i \in [1, m]$ such that $Q_{wi} \neq \emptyset$ and out of the first α processes from which p receives a response, at least $\lfloor \frac{|Q_{wi}|}{w_i+1} \rfloor + 1$ of them are in Q_{wi}.*

Intuitively, Assumption 2 requires that there are m sets of processes, the winning quorums, that answer faster than others, i.e., faster enough for subsets of these sets to be always included in every response set. In addition, every time a correct process issues a query, connectivity must allow for a subset of one of these winning quorums to receive and respond to the query. Note that winning quorums do not necessarily correspond to quorums returned by the failure detector at some point: instead they are sets of processes that have a tendency to be included in response sets.

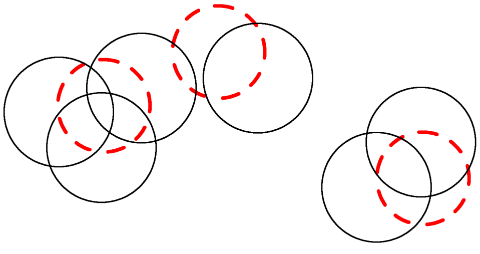

Fig. 1. Example of multiple winning quorums ($m = k = 3$).

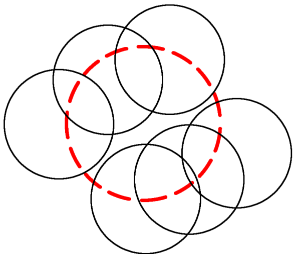

Fig. 2. Example of a single winning quorum ($m = 1, k = 3$).

The *weight* w_i of a winning quorum is a parameter which states which proportion of the winning quorum must be included in response sets. A winning quorum of weight 1 must be included in strict majority in a response set, whereas winning quorums of higher weights can be included in smaller proportions. The sum of all winning quorum weights is limited by k.

It is interesting to consider some extreme instances of this assumption. The first extreme is $m = k$. In this particular case, all winning quorums are necessarily of weight 1, and therefore each response set must include a strict majority of one of the winning quorums. Since each response set includes the strict majority of one out of k winning quorums, it is easy to see that out of any $k + 1$ response sets, at least two will necessarily intersect.

Fig. 1 shows an example for $m = k = 3$ in which winning quorums are represented by dashed red circles. Each solid black circle represents a response set. Note that out of any 4 response sets, at least 2 intersect.

Another extreme case is $m = 1$ and $w_1 = k$. In this particular case, all response sets must contain a small part of a single winning quorum.

Fig. 2 shows an example for $m = 1$ and $k = 3$. Similarly to Fig. 1, the winning quorum is represented by a dashed red circle, and response sets are represented by solid black circles. Once again, 2 out of any 4 response sets intersect.

The flexibility in the second example lies in which subset of the winning quorum will be included in each response set, while the flexibility in the first example lies in which winning quorum would be included in majority by each response set.

Assumption 2 implies that there is at least one winning quorum Q_{wi} such that at least $\lfloor \frac{|Q_{wi}|}{w_i+1} \rfloor + 1$ of the processes in Q_{wi} are correct. If $\alpha = \lfloor \frac{|Q_{wi}|}{w_i+1} \rfloor + 1 = 1$, there is no assumption on the number of failures but $|Q_{wi}| < w_i + 1$, which leaves minimal flexibility on the processes that must be included in every response set, thus strengthening the message pattern assumption. On the other hand, if $|Q_{wi}|$ (and therefore α) is high, the number of failures is limited but each response set must contain a subset of a larger set, which allows for more flexibility in the message pattern.

4.2.2 Assumption for Eventual Partial Leadership

In order to ensure the eventual partial leadership property, processes need to identify a local leader. Once again we choose to rely on a message pattern assumption. Since the eventual partial leadership property is supposed to be implemented on top of $\Sigma_{\perp,x}$, we can use the notion of quorum to define this new assumption.

Additionally, we use the order of processes in quorums to single out the leader. For this purpose, we assume that processes in a quorum are totally ordered. Any specific ordering can be used. A natural choice would be to use the order in which the processes were added to the quorum. Another simple choice would be to order according to process identifiers. For a process p_i and a quorum qr, if $p_i \in qr$, then we denote by $pos(p_i, qr)$ the position of p_i in qr according to the chosen total order. If p_i is the first process in qr_j^τ, then $pos(p_i, qr_j^\tau) = 1$. In this particular case, we say that p_i is the *candidate* of p_j at time τ.

We define potential eventual partial leaders as follows:

Definition 7 (Eventually winning process)**.** *A correct process p_l is called an eventually winning process if there is a time τ such that after $\tau, \forall \tau' \geq \tau, \forall p_i \in R_l \backslash \{p_l\}$:*
1) p_l is present in every quorum formed by p_i. $p_l \in qr_i^{\tau'}$.
2) p_l's identity is always positioned in p_i's quorum before the identities of other processes in R_i. $\forall p_j \in R_i \backslash \{p_l, p_i\}$: $pos(p_l, qr_i^{\tau'}) < pos(p_j, qr_i^{\tau'})$.
3) In every quorum formed by p_i, there is another process that also belongs to R_l. $\exists p_j \in R_l \backslash \{p_l, p_i\} : p_j \in qr_i^{\tau'}$.

Point (1) means that after some time, p_l must be fast enough to ensure that its responses arrive in time to take part in every local quorum.

The implication behind (2) depends on the chosen ordering method. If processes are ordered by date of addition to the quorum, then (2) implies that after some time, p_l must be faster than the rest of the recurrent neighborhood of p_i. If processes are ordered by process identities, p_l must have the smallest process identity in the recurrent neighborhood.

It is easy to see how (1) and (2) can be used: if p_l belongs to every quorum and is singled out by the quorum order, processes in R_l can reliably select their candidate as leader.

Note that (2) excludes the case $p_i = p_j$, since otherwise p_l would have to be placed before p_i in p_i's quorums. If processes are ordered by date of addition to the quorum, this expectation would be very unrealistic since receiving its own message is a local computation and should therefore be faster than receiving p_l's message.

Point (3) requires that processes in R_l must not only communicate with p_l but also with each other to some extent, which enables them to share the information that p_l is their candidate. Note that (3) also requires the processes p_l, p_i and p_j to be distinctly defined: therefore, in order for an eventually winning process to exist, there must be at least 3 correct processes in the system ($f \leq n - 3$). Since p_l, p_i and p_j must be included in the same quorums, α must also be equal to 3 or greater.

We call W the set of all eventually winning processes.

We can now formulate the assumption that will enable our failure detector algorithm to ensure the eventual partial leadership property:

Assumption 3 (Eventually winning γ-sources)**.** *For every correct process p_i, there is an eventually winning process p_l such that there is infinitely often a γ-journey from p_l to p_i.* $\forall p_i \in \mathcal{C}, \exists p_l \in W, \forall \tau : \exists \mathcal{J} \in \mathcal{J}^{\gamma}_{(p_l, p_i)} \wedge departure(\mathcal{J}) > \tau$.

Our $\Pi\Sigma_{\perp,x}$ algorithm will ensure that eventually winning processes are eventual partial leaders. As a result, this assumption will be sufficient to ensure the eventual partial leadership property.

4.3 Summary of Assumptions

Table 1 summarizes the assumptions presented in this section and the failure detector properties that rely on them for implementation.

TABLE 1
Assumptions for failure detector implementations

Assumption	Failure detector property
Assumption 1	$\Sigma_{\perp,k}$: quorum liveness $\Sigma_{\perp,k}$: quorum connectivity
Assumption 2	$\Sigma_{\perp,k}$: quorum intersection
Assumption 3	$\Pi\Sigma_{\perp,x}$: eventual partial leadership

The self-inclusion property of $\Sigma_{\perp,k}$ is absent from this table because it does not require any assumption and will simply be ensured through algorithmic properties.

4.4 Implementation of Message Pattern Assumptions

Assumptions 2 and 3 are very abstract and it can be difficult to judge at first glance how likely they are of being verified in a real network. This is because we attempt to isolate assumptions that are as close as possible to the minimum model strength required to ensure that our algorithm implements the $\Pi\Sigma_{\perp,x,y}$ failure detector. The message pattern model enables such an implementation while keeping our model generic and applicable to different networks. In this section we provide examples of more traditional assumptions that are sufficient to ensure Assumptions 2 and 3.

4.4.1 Implementation of Assumption 2

A simple and intuitive method is to assume that $|\mathcal{C}| \geq \lfloor \frac{n}{k+1} \rfloor + 1$. In this case, Assumption 2 is trivially verified with $m = 1, w_1 = k$ and $Q_{w1} = \Pi$. This implies that $\alpha \geq \lfloor \frac{n}{k+1} \rfloor + 1$, and, thus, the minimal size of quorums is sufficient to ensure intersection. This particular case is the method used to implement Σ_k in static networks in [10].

Another approach would be to use a partial synchrony assumption. For a given duration Δ, let us call Δ-journey a γ-journey $\mathcal{J}$ such that $arrival(\mathcal{J}) - departure(\mathcal{J}) \leq \Delta$. We then separate Π into two subsets: slow processes and fast processes. A slow process p_i is a process such that there is never a Δ-journey from p_i to any correct process $p_j \in \mathcal{C} \backslash \{p_i\}$. Fast processes are all other processes and Q is the set of all correct fast processes. The assumption is that for any correct process p_i and at any time, there are Δ-journeys linking p_i to at least $\lfloor \frac{|Q|}{k+1} \rfloor + 1$ processes from Q. Assumption 2 is verified with $m = 1, w_1 = k$ and $Q_{w1} = Q$.

4.4.2 Implementation of Assumption 3

One way to ensure Assumption 3 is that there is a correct subset Q of the system that is constantly connected and recognizes a leader $p_l \in Q$, that can reach the entire system infinitely often. The leader must be known from the other processes in Q from the start (it can simply be the lowest process identifier in Q, for example). When a process in Q issues a query, the communication layer for that process will then wait for a response from p_l and a response from another process in Q before delivering any other response. This is sufficient to ensure that p_l is the first process in every quorum formed in Q, and that processes in Q communicate with each other to a sufficient extent.

4.4.3 Practical issues

From a practical point of view, some types of networks are particularly adapted to ensure Assumptions 2 and 3. In wireless mesh networks ([21]), the nodes move around a fixed set of nodes and each mobile node eventually connects to a fixed node. Wireless sensor networks ([22]) can be organized in clusters; one node in each cluster is designated the cluster head. Messages sent between clusters are routed through the cluster heads of the sending and receiving clusters. An infra-structured mobile network ([12]) is composed of Mobile Hosts (MH) and Mobile Support Stations (MSS). A MH is connected to a MSS if it is located in its transmission range, and two MHs can communicate only through MSSs.

In each of these network models, there is a privileged subset of powerful nodes (fixed nodes, cluster heads, MSSs) that can be used as a winning quorum to satisfy Assumption 2 or as the neighborhood R_l of an eventual winning process p_l for Assumption 3.

Both assumptions can also be ensured from a probabilistic perspective. If a subset Q of the system is made of powerful nodes that respond to queries much faster than the rest of the nodes, then there is a high probability that Assumption 2 will be verified. Similarly, Assumption 3 can be verified in a probabilistic way with a leader that is simply a powerful process benefiting from very small communication delays with the processes around it.

5 FAILURE DETECTOR ALGORITHMS

In this section we first present a $\Sigma_{\perp,k}$ algorithm, then extend it to obtain a $\Pi\Sigma_{\perp,x,y}$ algorithm.

5.1 An Algorithm for $\Sigma_{\perp,k}$

Algorithm 1 implements the $\Sigma_{\perp,k}$ failure detector in unknown dynamic systems with asynchronous communications. It uses a query/response mechanism with round numbers in order to ensure quorum liveness.

5.1.1 Assumptions

Algorithm 1 implements $\Sigma_{\perp,k}$ in our model, provided that the following assumptions hold:

1) The system is a Time-Varying Graph of class 5-(α, γ) where α is the minimal size of a quorum and γ is the maximal time taken by a process to receive its own broadcasts (Assumption 1).

2) The run follows a generalized winning quorums message pattern (Assumption 2).

5.1.2 Notations

Each process p_i uses the following local variables:

r_i is the local round number of process p_i.

qr_i is the quorum currently returned by the failure detector for process p_i.

$recv\_from_i$ is the quorum buffer, containing all the identities of the processes whose response has been received by p_i since the time it last formed a new (complete) quorum. When the buffer contains enough information (i.e., at least α process identities), it becomes the new quorum and $recv\_from_i$ is reinitialized.

$last\_known_i$ is the knowledge p_i has of other processes round numbers. This variable and the associated mechanisms are not necessary for the correctness of the algorithm, they are simply used to improve performance by limiting the number of useless transmitted messages.

Process p_i calls the bcast(src, r_src, Q) primitive to broadcast a message to the processes currently in its neighborhood. A message contains the following values:

src is the identity of the original sender of the query (which is not necessarily the immediate sender of the message, since queries are forwarded multiple times).

r_src is the round number of src when this query was issued. Process src ignores responses to previous rounds.

Q is the set of the identities of processes who responded to the current query. When the query goes back to process src, it will add the content of this set to its quorum buffer.

Algorithm 1. *Implementation of $\Sigma_{\perp,k}$ for process p_i.*

```
init
    $r_i \leftarrow 0$ //Local round number
    $qr_i \leftarrow \perp$ //The quorum returned by $\Sigma_{\perp,k}$ for $p_i$
    $recv\_from_i \leftarrow \{p_i\}$ //Quorum buffer
    $last\_known_i \leftarrow \emptyset$ //Round numbers of known processes
    bcast($p_i, 0, \emptyset$)

upon reception of ($src$, $r\_src$, $Q$)from $p_j$ do
    if $src = p_i$ and $r\_src = r_i$ then //Response
        $recv\_from_i \leftarrow recv\_from_i \cup Q$
        if $|recv\_from_i| \geq \alpha$ then
            $qr_i \leftarrow recv\_from_i$
            $recv\_from_i \leftarrow \{p_i\}$
            $r_i \leftarrow r_i + 1$
        bcast($p_i, r_i, \emptyset$)
    else if $src \neq p_i$ then //Query
        if $\exists last\_r \mid \langle src, last\_r \rangle \in last\_known_i$
                $\wedge\, last\_r \leq r\_src$ then
            $last\_known_i \leftarrow last\_known_i \backslash \{\langle src, last\_r \rangle\}$
            $last\_known_i \leftarrow last\_known_i \cup \{\langle src, r\_src \rangle\}$
            bcast($src, r\_src, Q \cup \{p_i\}$)
        else if $\langle src, - \rangle \notin last\_known_i$ then
            $last\_known_i \leftarrow last\_known_i \cup \{\langle src, r\_src \rangle\}$
            bcast($src, r\_src, Q \cup \{p_i\}$)
        else
            do nothing
```

5.1.3 Algorithm Description

The principle behind the algorithm is the following: every process p_i keeps broadcasting queries for round r_i until it receives enough responses to form a quorum of size at least α, then it increments r_i and proceeds with the next round.

Contrarily to most query/response algorithms, Algorithm 1 only uses one type of messages. A message is both a query and a response, depending on which process receives it. Every message travels from process to process, until it goes back to the original message sender. If the test on line 8 is true, the message is considered as a response to the current round query. If instead the test on line 15 is true, the message is considered as a query from another process.

Every process identity received in a response for the current round is added to the $recv\_from_i$ buffer (line 9), and when the buffer size gets superior or equal to α, then a new quorum is formed by copying $recv\_from_i$ into qr_i and resetting the buffer (lines 10 – 13).

If a received message is a query from another process, p_i updates its local knowledge and then adds its own identity to the message and rebroadcasts it unless another query for a higher round has been previously received from the same emitter (lines 15 – 25).

At first glance it might look like process p_i only broadcasts its queries once (lines 6 and 14), but keep in mind that processes receive their own broadcasts. Therefore, after initially broadcasting a new query, p_i will receive it at most γ instants later and broadcast it again (line 14).

The same rebroadcasting approach applies for queries from other processes. Once p_i has received a message from src for round r_src, it will keep rebroadcasting it (lines 20 and 23) until it is informed that src moved on past round r_src (the test on lines 16 – 17).

Based on the assumption of generalized winning quorums, the only action necessary to ensure quorum intersection is to make sure that quorums are formed from at least α process identities, which is guaranteed by line 10.

Quorum liveness is ensured because (1) correct processes keep forming new quorums from fresh information infinitely often thanks to class 5-(α, γ) and (2) the identities of crashed processes are excluded from new quorums since the r_src in their responses are eventually outdated (line 8).

5.1.4 Proof of Correctness

Lemma 1. *In a TVG of class 5-(α, γ) where Assumption 2 holds, Algorithm 1 ensures the quorum intersection property of $\Sigma_{\perp,k}$.*

Proof: Assumption 2 implies that $\sum_{i=1}^{m} w_i \leq k$. For any number $w \in [1, k]$, we denote n_w the number of winning quorums of weight w. It follows that $\sum_{w=1}^{k} w \times n_w \leq k$.

Additionally, Assumption 2 imposes that every response set includes responses from a winning quorum Q_{wi} of weight w_i such that at least $\lfloor \frac{|Q_{wi}|}{w_i+1} \rfloor + 1$ processes from Q_{wi} are part of that response set. It follows that, if $w_i + 1$ response sets are formed from the same winning quorum Q_{wi}, at least two of these response sets intersect.

If no two response sets are to intersect, then at most w_i response sets can be formed from a given winning quorum Q_{wi}. Therefore, for any number $w \in [1, k]$, at most $w \times n_w$ response sets can be formed from the set of all winning quorums of weight w. It follows finally that at most $\sum_{w=1}^{k} w \times n_w$ response sets can be formed from the set of all winning quorums without any two of them intersecting. Since $\sum_{w=1}^{k} w \times n_w \leq k$, at least two out of any $k + 1$ response sets intersect.

Lines 10 and 11 of Algorithm 1 ensure that quorums include the first α responses (response set) to the current query. Therefore every quorum includes a response set, and the quorum intersection property of $\Sigma_{\perp,k}$ is ensured. □

Lemma 2. *In a TVG of class 5-(α, γ), every correct process executing Algorithm 1 forms a new quorum infinitely often.*

Proof: Since it uses a query-response mechanism, Algorithm 1 requires every correct process to reach and be reached back by α processes, which is ensured by a TVG of class 5-(α, γ) infinitely often. Even if a journey includes waiting time during which the process holding the message is isolated, the process keeps memory of the message by rebroadcasting it to itself, and transmits it to other processes as soon as it it stops being isolated. As a result, every correct process will receive responses from α processes infinitely often, and therefore pass the test on line 10 infinitely often. □

Lemma 3. *In a TVG of class 5-(α, γ), Algorithm 1 ensures the quorum liveness property of $\Sigma_{\perp,k}$.*

Proof: By definition, faulty processes will crash or leave the system forever in a finite time. Let $t \in \mathcal{T}$ be the time at which the last faulty process crashes or leaves the system forever. Since $f < n$, there are correct processes in the system. Lemma 2 ensures that each of these processes forms a new quorum sometime after t. Let $\tau \in \mathcal{T}$ be a time such that $\tau > t$ and every remaining process has formed a *quorum* between t and τ. Therefore, every quorum being currently built at τ has been started after t, which means no faulty process can possibly respond to the corresponding query message. As a result, every new quorum formed after τ contains only correct processes. It follows that Algorithm 1 ensures the quorum liveness property of $\Sigma_{\perp,k}$. □

Lemma 4. *In a TVG of class 5-(α, γ), Algorithm 1 ensures the quorum connectivity property of $\Sigma_{\perp,k}$.*

Proof: The properties of a TVG of class 5-(α, γ) ensure that every correct process will always receive enough messages to pass the test on line 10 and keep forming new quorums infinitely often. The test on line 8 ensures that processes only form quorums from messages from the current round. It follows that eventually, every correct process p_i only includes in its quorums processes which receive its queries and respond to it infinitely often. Therefore, p_i can send and receive messages infinitely often to and from the processes that are infinitely often in its quorums.

Let $p_i \in \mathcal{C}$ and $p_j \in R_i$. By definition of R_i, p_i and p_j's quorums intersect infinitely often and thus there must exist a correct process p_m such that p_m is infinitely often in p_i's quorums and p_m is infinitely often in p_j's quorums. As a result, p_m can receive messages from p_i infinitely often and p_j can receive messages from p_m infinitely often. Therefore if messages are routed through p_m, p_j can receive messages from p_i infinitely often. □

Theorem 1. *In a TVG of class 5-(α, γ) where Assumption 2 holds, Algorithm 1 implements a $\Sigma_{\perp,k}$ failure detector.*

Proof: It follows from Lemmas 1, 3 and 4 that the algorithm ensures the quorum intersection, quorum liveness and quorum connectivity properties respectively.

Self-inclusion is ensured by the fact that every quorum is formed from the buffer $recv\_from_i$ (line 11), and the buffer is always initialized with p_i (lines 4 and 12). □

5.2 An Algorithm for $\Pi\Sigma_{\perp,x}$

Algorithm 2 is an extension of Algorithm 1 aiming at implementing $\Pi\Sigma_{\perp,x}$ in our dynamic model. It adds an election mechanism to the original algorithm in order to identify an eventual partial leader.

This leader election mechanism relies on the quorum order, as defined in Section 4.2.2. Every time a process forms a new quorum, it selects the first process in the quorum as candidate for the leader election. If a process is the candidate of every other process in its quorum, then it selects itself as leader; otherwise it selects its candidate as leader.

Algorithm 2. *Implementation of* $\Pi\Sigma_{\perp,x}$ *for process* p_i.

```
1:  init
2:      $r_i \leftarrow 0$ //Local round number
3:      $qr_i \leftarrow \perp$ //The quorum returned by $\Pi\Sigma_{\perp,x}$ for $p_i$
4:      $recv\_from_i \leftarrow \{p_i\}$ //Quorum buffer
5:      $last\_known_i \leftarrow \emptyset$ //Round numbers of known processes
6:      $leader_i \leftarrow p_i$ //The leader returned by $\Pi\Sigma_{\perp,x}$ for $p_i$
7:      $candidate_i \leftarrow \perp$ //$p_i$'s current candidate for leadership
8:      $candidates_i \leftarrow \emptyset$ //Candidates of processes in $recv\_from_i$
9:      bcast$(p_i, 0, \emptyset, \emptyset)$

10: upon reception of ($src$, $r\_src$, $Q$, $cands$)from $p_j$ do
11:     if $src = p_i$ and $r\_src = r_i$ then //Response
12:         $recv\_from_i \leftarrow recv\_from_i \cup Q$
13:         $candidates_i \leftarrow candidates_i \cup cands$
14:         if $|recv\_from_i| \geq \alpha$ then
15:             $qr_i \leftarrow recv\_from_i$
16:             $recv\_from_i \leftarrow \{p_i\}$
17:             $r_i \leftarrow r_i + 1$
18:             $candidate_i \leftarrow p_l \mid (pos(p_l, qr_i) = 1 \wedge p_l \neq p_i)$
19:                 $\vee(pos(p_i, qr_i) = 1 \wedge pos(p_l, qr_i) = 2)$
20:             if $candidates_i = \{p_i\}$ or $\emptyset$ then
21:                 $leader_i \leftarrow p_i$
22:             else
23:                 $leader_i \leftarrow candidate_i$
24:             $candidates_i \leftarrow \emptyset$
25:         bcast$(p_i, r_i, \emptyset, \emptyset)$
26:     else if $src \neq p_i$ then //Query
27:         if $\exists last\_r \mid \langle src, last\_r\rangle \in last\_known_i$
28:                 $\wedge\, last\_r \leq r\_src$ then
29:             $last\_known_i \leftarrow last\_known_i \backslash \{\langle src, last\_r\rangle\}$
30:             $last\_known_i \leftarrow last\_known_i \cup \{\langle src, r\_src\rangle\}$
31:             bcast$(src, r\_src, Q \cup \{p_i\}, cands \cup \{candidate_i\})$
32:         else if $\langle src, -\rangle \notin last\_known_i$ then
33:             $last\_known_i \leftarrow last\_known_i \cup \{\langle src, r\_src\rangle\}$
34:             bcast$(src, r\_src, Q \cup \{p_i\}, cands \cup \{candidate_i\})$
35:         else
36:             do nothing
```

5.2.1 Assumptions

Algorithm 2 implements $\Pi\Sigma_{\perp,x}$ in our model, provided that the following assumptions hold:

1) The system is a Time-Varying Graph of class 5-(α, γ) where α is the minimal size of a quorum and γ is the maximal time taken by a process to receive its own broadcasts (Assumption 1).

2) The run follows a generalized winning quorums message pattern (Assumption 2).

3) The system verifies the eventually winning γ-sources assumption (Assumption 3).

5.2.2 Notations

Algorithm 2 uses the same notations as Algorithm 1. Additionally, each process p_i uses the following local variables:

$leader_i$ is the leader returned by the failure detector for process p_i. $leader_i$ is initially p_i, and is later updated on lines 21 or 23.

$candidate_i$ is the first process in p_i's most recent quorum (excluding p_i itself). It is affected in lines 18 – 19. $candidate_i$ is initialized to $\perp$ and is added to sets (lines 31 and 34). We take the convention that $\emptyset \cup \{\perp\} = \emptyset$.

$candidates_i$ is the set of the candidates of the processes in $recv\_from_i$ (except p_i). p_i will only elect itself as leader (line 21) if $candidates_i$ only contains p_i (i.e., p_i is the candidate of every process in $recv\_from_i \backslash \{p_i\}$) or if $candidates_i$ is empty (i.e., p_i considers itself alone).

In addition to the message parameters described for Algorithm 1, messages sent by processes contain the $cands$ parameter, which is the set of the candidates of the processes in Q, at the time when they responded to the query. It carries the information necessary for process p_i to build its $candidates_i$ set on line 13.

5.2.3 Algorithm Description

Algorithm 2 follows the same structure and uses the same mechanisms to build quorums as Algorithm 1. Its additional code aims to select partial leaders according to the eventual partial leadership property of $\Pi\Sigma_{\perp,x}$. The extension added to Algorithm 1 is composed of two parts: candidate selection and leader selection.

Candidate selection revolves around the notion of quorum order presented in Section 4.2.2. The first process in every quorum is selected as the candidate. Whenever a process p_i completes a new quorum (meaning it passes the test on line 14), it handles the end of the round similarly to Algorithm 1 (lines 15 – 17). It then identifies the first process in the new quorum (excluding itself) according to the chosen ordering method in lines 18 – 19 and selects it as its $candidate_i$. If it was possible for p_i to be its own candidate, and if quorums were ordered by date of response, then p_i would always be its own candidate.

By virtue of Assumption 3, an eventually winning process p_l will eventually be forever the candidate of every process in $R_l \backslash \{p_l\}$. However, p_l cannot be its own candidate. Therefore, information about p_l's own quorum order is not sufficient for p_l to select itself as the leader. It must take into account the candidates of other processes.

This is the purpose of the $candidates_i$ variable. Other processes inform p_i of their respective candidates by including it in their responses (lines 31 and 34), and p_i gathers this information in $candidates_i$ in line 13. When p_i completes a quorum, $candidates_i$ contains the candidates of the processes currently in $qr_i \backslash \{p_i\}$.

If every process in qr_i agrees on p_i as the candidate (or if p_i is the only process in qr_i), then p_i selects itself as the leader (line 21). Otherwise, p_i selects $candidate_i$ (line 23).

Note that point (3) of Definition 7 prevents the problematic case where a process p_i only includes in its quorums an eventually winning process p_l and processes in $\Pi \backslash R_l$. In this scenario, it would be possible for every process in R_i (including p_l) to chose p_i as its candidate, thus misleading p_i into selecting itself as the leader infinitely often.

5.2.4 Proof of Correctness

We should prove that, if Assumptions 1, 2 and 3 hold, then Algorithm 2 ensures the 5 properties of $\Pi\Sigma_{\perp,x}$.

Lemma 5. *In a TVG of class 5-(α, γ) where Assumption 2 holds, Algorithm 2 ensures the self-inclusion, quorum intersection, quorum liveness and quorum connectivity properties of* $\Pi\Sigma_{\perp,x}$.

Proof: The added code from Algorithm 1 does not modify the way the qr_i variable is initialized and updated. Therefore, the proof for Theorem 1 holds for Algorithm 2. □

Lemma 6. *Every eventually winning process* p_l *is eventually forever the* $candidate_i$ *of every process* $p_i (\neq p_l)$ *of its recurrent neighborhood.* $\forall p_l \in W, \forall p_i \in R_l \backslash \{p_l\} : \exists \tau : \forall \tau' \geq \tau : candidate_i = p_l$ *at time* τ'.

Proof: It follows from the properties of a TVG of class 5-(α, γ) that correct processes will keep passing the test on line 14, and therefore form new quorums infinitely often.

By contradiction, let us assume the following: $\exists p_l \in W, \exists p_i \in R_l \backslash \{p_l\}, \exists p_m \in \Pi \backslash \{p_l\}, \forall \tau : \exists \tau' \geq \tau : candidate_i = p_m$ at time τ'. There are, thus, two cases:

$\boldsymbol{p_m \notin R_i}$. By definition of R_i, there is a time after which p_i's quorums never intersect with p_m's quorum. By construction of the algorithm (lines 4 and 16), self-inclusion is ensured (every process belongs to its own quorums). Thus, there is a time after which p_m is never in p_i's quorums, and therefore it can never be selected as $candidate_i$ on lines 18 – 19 after this time.

$\boldsymbol{p_m \in R_i}$. Since p_l is an eventually winning process, there is a time after which (1) p_l is in every quorum formed by p_i and (2) in every quorum formed by p_i that includes p_m, p_l is positioned before p_m. As a result, p_m can never be selected as $candidate_i$ on lines 18 – 19 after this time. □

W is the set of all eventually winning processes, and L is the set of all eventual partial leaders.

Lemma 7. *Every eventually winning process is an eventual partial leader.* $W \subseteq L$.

Proof: Let $p_l \in W$. p_l is an eventual partial leader if and only if, for every $p_i \in R_l$, eventually $leader_i = p_l$ forever. There are two cases:

$\boldsymbol{p_i = p_l}$. It follows from the definition of R_l and from self-inclusion that there is a time after which every process that is not in R_l will stop appearing in the quorums formed by p_l. It follows that there is a time τ_1 such that $\forall \tau_1' > \tau_1, qr_l^{\tau_1'} \subseteq R_l$. If $\alpha = 1$, then $qr_l^{\tau_1'} = \{p_l\}$ and therefore $candidates_l = \emptyset$ at time τ_1' (by construction of $candidates_l$). If $\alpha > 1$, since the definition of R_l is symmetrical, $\forall \tau_1' > \tau_1, \forall p_j \in qr_l^{\tau_1'} : p_l \in R_j$. It then follows from Lemma 6 that $\exists \tau_2 \geq \tau_1, \forall \tau_2' > \tau_2, \forall p_j \neq p_l \in qr_l^{\tau_2'} : candidate_j = p_l$ at time τ_2'. Since p_l will keep forming new quorums with fresh information, $\exists \tau_3 \geq \tau_2$ such that every time after τ_3 that p_l completes a round, then $candidates_l = \{p_l\}$. As a result, after time τ_3, p_l will always pass the test on line 20 and, therefore, will forever identify itself as the leader.

$\boldsymbol{p_i \neq p_l}$. According to point (3) of the eventually winning process definition, $\exists \tau_1, \forall \tau_1' > \tau_1, \exists p_j \in R_l : p_j \in qr_i^{\tau_1'}$. It follows from Lemma 6 that $\exists \tau_2 \geq \tau_1, \forall \tau_2' > \tau_2, candidate_j = candidate_i = p_l$ at time τ_2'. Since p_i will keep forming new quorums with fresh information received from p_j, $\exists \tau_3 \geq \tau_2$ such that every time after τ_3 that p_l completes a round, then $p_l \in candidates_i$. As a result, after τ_3, p_i will always fail the test on line 20 and will forever identify $candidate_i = p_l$ as the leader.

In both cases, p_i selects p_l as leader forever, which makes p_l an eventual partial leader. □

Lemma 8. *If the eventually winning* γ*-sources assumption (Assumption 3) holds, then Algorithm 2 ensures the eventual partial leadership property of* $\Pi\Sigma_{\perp,x}$.

Proof: It follows from Assumption 3 that $\forall p_i \in \mathcal{C}, \exists p_l \in W, \forall \tau : \exists \mathcal{J} \in \mathcal{J}^{\gamma}_{(p_l,p_i)} \wedge departure(\mathcal{J}) > \tau$. It follows from Lemma 7 that $p_l \in L$. Since we assume fair-lossy channels, then if p_l sends messages infinitely often, then p_i will receive messages from p_l infinitely often. □

Theorem 2. *In a TVG of class 5-(α, γ) where Assumptions 2 and 3 hold, Algorithm 2 implements a* $\Pi\Sigma_{\perp,x}$ *failure detector.*

Proof: Follows directly from Lemmas 5 and 8. □

5.3 An Algorithm for $\Pi\Sigma_{\perp,x,y}$

An algorithm for $\Pi\Sigma_{\perp,x,y}$ simply consists in executing y instances of Algorithm 2 simultaneously. This algorithm relies on Assumptions 1 , 2 and 3. However, Assumption 3 is only required to apply for one out of the y instances of the algorithm.

6 A k-SET AGREEMENT ALGORITHM

In [9], the authors proposed an algorithm for k-set agreement using $\Pi\Sigma_{x,y}$ for static networks. The k-set algorithm itself is very simple. It only deals with the liveness property of k-set agreement (termination) and encapsulates the safety properties (validity and agreement) into the $Alpha_x$ sub protocol. In this section we will adapt the $Alpha_x$ and k-set agreement algorithms for dynamic networks.

6.1 The $Alpha_x$ Sub Protocol

$Alpha$ was introduced in [23] as a way to exactly capture the safety properties of consensus (that is, validity and agreement). It is thus complementary to the Ω failure detector, which is necessary to ensure liveness (the termination property). $Alpha$ was later generalized in [24] into KA for the k-set agreement problem.

In [9], Mostéfaoui, Raynal and Stainer define $Alpha_x$ as an extended, weaker version of the KA of [24]. $Alpha_x$ is a distributed object used to store values proposed by processes. It initially stores the default value $\perp$. It provides processes with an operation $Alpha.propose_x(r, v)$ that returns a value (possibly $\perp$). The round number r is a logical time and v is a proposed value. It is assumed that (a) each process will use increasing round numbers in successive invocations of $Alpha.propose_x()$ and (b) distinct processes use different round numbers. An $Alpha_x$ object is defined by the following properties:

Termination. Any invocation of $Alpha.propose_x()$ by a correct process terminates.

Validity. If $Alpha.propose_x(r, v)$ returns $v' \neq \perp$, then $Alpha.propose_x(r', v')$ has been invoked with $r' \leq r$.

Quasi-agreement. At most x different non-$\perp$ values can be returned by different $Alpha.propose_x()$ invocations.

Obligation. Let p_l be a correct process and $Q(l, \tau) = \{p_i \in \mathcal{C} | \forall \tau_i, \tau_l \geq \tau : qr_i^{\tau_i} \cap qr_l^{\tau_l} = \emptyset\}$. If, after time τ, (a) only p_l and processes of $Q(l, \tau)$ invoke $Alpha.propose_x()$

and (b) p_l invokes $Alpha.propose_x()$ infinitely often, then at least one invocation issued by p_l returns a non-$\bot$ value.

Note that the termination property of $Alpha_x$ is not related to the termination property of the k-set agreement.

In order to ensure the safety properties of k-set agreement, it is not necessary to make use of the eventual partial leadership property of $\Pi\Sigma_{\bot,x}$ and therefore, the $Alpha_x$ algorithm presented here does not make use of the $leader_i$ variable. However, the k-set agreement algorithm implements the termination property of k-set agreement by relying on the obligation property of $Alpha_x$ and the eventual partial leadership property of $\Pi\Sigma_{\bot,x}$.

The definitions in [10] and [9], that propose k-set agreement algorithms for static networks, use different obligation properties. The $Alpha_x$ in this paper is the one defined in [9], which is weaker than the one in [10] by being Σ_x-aware.

6.2 $Alpha_x$ Algorithm

In this section we propose an algorithm implementing $Alpha_x$ for our model enriched with $\Pi\Sigma_{\bot,x}$, adapted from the algorithm in [9].

The algorithm gives each proposed value a priority. Each process p_i keeps a value est_i, which is its current estimation of the value it will decide, and a pair (lre_i, pos_i) which defines the priority of value est_i. lre_i is the highest round seen by p_i and pos_i is the position of value est_i within round lre_i. The position is used to fix priority on proposed values.

The function $g(\rho, \delta) = 2^{\delta}(\rho - 1) + 1$ where ρ is the position of value v on round r and $\delta = r' - r$, with $r' \geq r$, is used to compute the position of v on round r'.

If value v has priority ρ at round r and value v' has priority ρ' at round r' with $r \leq r'$, v has lower priority than v' at round r' if and only if $g(\rho, r' - r) < \rho'$ or $(g(\rho, r' - r) = \rho') \wedge (v < v')$.

The $Alpha.propose_x()$ function is composed of two phases. In the read phase (lines 6 – 14), the process attempts to gather knowledge on the values proposed by other processes in a quorum (as defined by $\Sigma_{\bot,x}$) by sending REQ_R messages and receiving RSP_R messages. If a process in the quorum is already computing a higher round, p_i returns $\bot$ (line 11). Otherwise, it selects the highest priority value it knows of (lines 12 – 13), and proceeds to the write phase.

In the write phase (lines 15 – 24), the process attempts to raise the priority of its current estimated value by communicating it to other processes in a quorum with REQ_W messages and receiving RSP_W messages. Once again, if any process in the quorum is computing a higher round, p_i returns $\bot$ (line 22). If another process has a value of higher priority for the current round, p_i adopts it as its new estimated value (lines 23 – 24). p_i then raises pos_i by 1 (line 16) and repeats the write phase until it manages to raise a value to position 2^r (line 15) or until it encounters a process in a higher round (line 22).

The following modifications were made to the original algorithm in [9] in order for the algorithm to ensure the properties of $Alpha_x$ in dynamic networks:

The original algorithm assumed a complete, static communication graph with reliable channels and therefore every message was only sent once. In our model we need messages to be rebroadcast (lines 31, 33, 42 and 44). This mechanism ensures that (1) the emitting process will rebroadcast its own message every γ units of time; and (2) the reception of the message will not be restricted to the neighbors of the emitting process. The message will be received by every process that can be reached through a γ-journey.

Since messages are rebroadcast, the direct emitter of a message is not necessarily the source of the message. For this reason, we added the process identifier of the responding process in message types RSP_R and RSP_W.

Algorithm 3. *Implementation of $Alpha_x$ using $\Pi\Sigma_{\bot,x}$ in dynamic networks for process p_i.*

```
1:  init
2:      $lre_i \leftarrow 0$ //The last round entered by $p_i$
3:      $est_i \leftarrow \bot$ //The value that $p_i$ currently plans on deciding
4:      $pos_i \leftarrow 0$ //The position of $est_i$ within round $lre_i$

5:  function ALPHA.PROPOSE$_X$($r$,$v_i$)
6:      repeat $Q_i \leftarrow qr_i$; bcast REQ_R($r, Q_i$)
7:      until $Q_i \neq \bot$ and $\forall p_j \in Q_i$ :
8:          RSP_R($r, p_j, \langle lre_j, pos_j, val_j \rangle$) received
9:      $rcv_i \leftarrow \{\langle lre_j, pos_j, est_j \rangle : p_j \in Q_i \wedge$
10:         RSP_R($r, p_j, \langle lre_j, pos_j, est_j \rangle$) received$\}$
11:     if $\exists \langle lre, -, - \rangle \in rcv_i : lre > lre_i$ then return($\bot$)
12:     $pos_i \leftarrow \max\{pos \mid \langle r, pos, v \rangle \in rcv_i\}$
13:     $est_i \leftarrow \max\{v \mid \langle r, pos_i, v \rangle \in rcv_i\}$
14:     if $est_i = \bot$ then $est_i \leftarrow v_i$
15:     while $pos_i < 2^r$ do
16:         $pos_i \leftarrow pos_i + 1$; $pst_i \leftarrow pos_i$
17:         repeat $Q_i \leftarrow qr_i$; bcast REQ_W($r, pst_i, est_i, Q_i$)
18:         until $Q_i \neq \bot$ and $\forall p_j \in Q_i$ :
19:             RSP_W($r, pst_i, p_j, \langle lre_j, pos_j, val_j \rangle$) received
20:         $rcv_i = \{\langle lre_j, pos_j, est_j \rangle : p_j \in Q_i \wedge$
21:             RSP_W($r, pst_i, p_j, \langle lre_j, pos_j, est_j \rangle$) received$\}$
22:         if $\exists lre : \langle lre, -, - \rangle \in rcv_i : lre > lre_i$ then return($\bot$)
23:         $pos_i \leftarrow \max\{pos \mid \langle r, pos, v \rangle \in rcv_i\}$
24:         $est_i \leftarrow \max\{v \mid \langle r, pos_i, v \rangle \in rcv_i\}$
25:     return($est_i$)

26: upon reception of REQ_R($rd, Q$) do
27:     if $p_i \in Q$ then
28:         if $rd > lre_i$ then
29:             $pos_i \leftarrow$ g($pos_i, rd - lre_i$); $lre_i \leftarrow rd$
30:         bcast RSP_R($rd, p_i, \langle lre_i, pos_i, est_i \rangle$)
31:     bcast REQ_R($rd, Q$)

32: upon reception of RSP_R($rd, p_j, \langle lre_j, pos_j, est_j \rangle$) do
33:     bcast RSP_R($rd, p_j, \langle lre_j, pos_j, est_j \rangle$)

34: upon reception of REQ_W($rd, pos, est, Q$) do
35:     if $p_i \in Q$ then
36:         if $rd \geq lre_i$ then
37:             $pos_i \leftarrow$ g($pos_i, rd - lre_i$); $lre_i \leftarrow rd$
38:             if $pos > pos_i$ then $est_i \leftarrow est$; $pos_i \leftarrow pos$
39:             else if $pos = pos_i$ then
40:                 $est_i \leftarrow \max\{est_i, est\}$
41:         bcast RSP_W($rd, pos, p_i, \langle lre_i, pos_i, est_i \rangle$)
42:     bcast REQ_W($rd, pos, est, Q$)

43: upon reception of RSP_W($rd, pos, p_j, \langle lre_j, pos_j, est_j \rangle$) do
44:     bcast RSP_W($rd, pos, p_j, \langle lre_j, pos_j, est_j \rangle$)
```

The original algorithm uses a selective multicast for both the read and write phases, i.e., messages are sent only to the processes in a quorum Q_i. Our algorithm uses broadcasts as defined in Section 2 (lines 6 and 17) and transmits Q_i with the message. All receiving processes will rebroadcast the message, but only the processes within Q_i will deliver it (lines 27 and 35).

Theorem 3. *In our model augmented with $\Pi\Sigma_{\perp,x}$, Algorithm 3 ensures the properties of $Alpha_x$.*

Proof. The modifications added to the original algorithms from [10] and [9] do not allow the algorithm to add new values, therefore the proof for validity in the original papers holds. Similarly, the proofs for obligation in [9] and quasi-agreement in [10] do not rely on any static connectivity assumption, and instead rely on algorithm behavioural properties which were not altered in our version. Therefore, the original proofs hold for Algorithm 3.

Concerning termination, the only possibility for an invocation not to terminate is that process p_i waits forever for a response message in one of the repeat loops (lines 6–8 and 17–19). Let us assume by contradiction that p_i waits forever for responses. The liveness property of $\Pi\Sigma_{\perp,x}$ ensures that eventually p_i only sends queries to correct processes and waits for responses from correct processes. Given that the set of correct processes is finite, the set of possible correct quorums is finite too. It follows that there is a correct quorum Q such that infinitely often, $qr_i = Q$, and therefore according to the quorum connectivity and self-inclusion properties of $\Pi\Sigma_{\perp,x}$, there are recurrent journeys between any process in Q and p_i. As a result, all the processes from Q will eventually receive the queries from p_i, and p_i will eventually receive the responses from the processes in Q, and, therefore, exit the repeat loop. ☐

6.3 k-Set Agreement Algorithm

Given an $Alpha_x$ object and a $\Pi\Sigma_{\perp,x,y}$ failure detector, solving k-set agreement is simple. The algorithm given here is an adaptation of the one given in [9] for dynamic networks. We first solve x-set agreement with $\Pi\Sigma_{\perp,x}$ (Algorithm 4), and then k-set agreement with $\Pi\Sigma_{\perp,x,y}$ for $k \geq xy$.

Algorithm 4. *x-Set agreement with $Alpha_x$ using $\Pi\Sigma_{\perp,x}$ in dynamic networks for process p_i.*

```
1: init
2:     $dec_i \leftarrow \perp$ //The value decided by $p_i$ ($\perp$ if $p_i$ has not decided)
3:     $prime_i \leftarrow$ the $i^{th}$ prime number //Constant
4:     $r_i \leftarrow prime_i$ //The current round number

5: function PROPOSE($v_i$)
6:     while $dec_i = \perp$ do
7:         if $leader_i = p_i$ then
8:             $dec_i \leftarrow Alpha.propose_x(r_i, v_i)$
9:             $r_i \leftarrow r_i \times prime_i$
10:    decide($dec_i$)
11:    bcast DECISION($dec_i$)

12: upon reception of DECISION($d$) do
13:    if $dec_i = \perp$ then
14:        $dec_i \leftarrow d$
15:        decide($d$)
16:    bcast DECISION($d$)
```

A well formed invocation of $Alpha.propose_x(r, v)$ is an invocation such that two processes cannot use the same round number r, and successive round numbers for a given process are increasing. To this end, each process p_i initially computes $prime_i$, the i^{th} prime number. p_i then uses $prime_i$ as its first round number, and multiplies it by $prime_i$ after every round. As a result, the round number of p_i increases and is always a power of $prime_i$, which ensures that two distinct processes always use distinct round numbers.

Theorem 4. *In our model augmented with $\Pi\Sigma_{\perp,x}$ and with an $Alpha_x$ object, Algorithm 4 solves the x-set agreement problem.*

Proof. The test on line 6 ensures that the $\perp$ value is never decided. From this point on, the validity of the $Alpha_x$ object is enough to ensure the validity of x-set agreement. Similarly, the quasi-agreement property of $Alpha_x$ is enough to ensure the agreement property of x-set agreement.

The eventual partial leadership property of $\Pi\Sigma_{\perp,x}$ ensures that if every leader in L decides, then eventually every correct process will receive a DECISION message from a process in L. As a result, the proof for the termination property provided in [9] holds for Algorithm 4. ☐

Similarly to [9], a simple k-set algorithm can be obtained by running y instances of Algorithm 4, the j^{th} one ($1 \leq j \leq y$) relying on the component $FD_i[j]$ of failure detector $\Pi\Sigma_{\perp,x,y}$ for every process p_i. A process decides the same value decided by the first of the y instances that terminates. As there are y instances of the algorithm and each of them can decide x values at most, it follows that at most xy values can be decided. Therefore, the algorithm solves k-set agreement for $k \geq xy$.

7 RELATED WORK

In this section, we will first present a number of articles that offer solutions to agreement problems in dynamic systems, then we will compare the assumptions we use in this paper with existing models in the literature.

7.1 Agreement in Dynamic Systems

A number of papers have proposed solutions to agreement problems in dynamic networks, while relying on various timeliness, failure pattern, and connectivity assumptions.

The synchronous model is the most widely used to solve dynamic consensus in the literature. In [3], Kuhn et al. consider a model with a fixed set of processes communicating in synchronous rounds, and propose algorithms solving consensus, simultaneous consensus (all processes decide within the same round), and Δ-coordinated consensus (all processes decide within Δ rounds of each other). Biely et al. provide another algorithm for consensus in a similar model in [1], with weaker connectivity assumptions. In order to better formulate timeliness assumptions in the Time-Varying Graph formalism of [11], Gómez-Calzado et al. introduce in [19] the notion of timely journeys. They also propose an algorithm solving the Terminating Reliable Broadcast, which is equivalent to consensus in their synchronous model.

Fewer papers have studied asynchronous dynamic consensus. In [25], Taheri and Izadi propose a protocol solving the stronger problem of Byzantine consensus in an asynchronous dynamic system, using the necessary assumption that no more than $\lfloor \frac{n-1}{3} \rfloor$ processes are faulty. Benchi et al. also provide in [5] an algorithm for asynchronous dynamic consensus under a similar failure pattern assumption.

From a failure detector perspective, some papers chose to implement the eventual leader detector Ω [7] (the weakest failure detector to solve consensus in message passing environments with a majority of correct processes) in dynamic systems as a step towards consensus. Cao et al. in [12] study eventual leadership in dynamic systems, proposing a model in which the system is composed of two sets of nodes: fixed support stations forming a static complete graph with asynchronous communications, and mobile hosts communicating through the support stations. Eventual leader protocols for dynamic networks were also proposed by Gómez-Calzado et al. in [26] using partial synchrony assumptions, and by Arantes et al. in [27] using message pattern assumptions in a timer-free model.

To the best of our knowledge, only three papers have studied the problem of k-set agreement in dynamic systems. Biely et al. in [2] presented an algorithm for gracefully degrading consensus in synchronous dynamic networks. The algorithm solves consensus if the network conditions allow for it, and falls back on solving k-set agreement, otherwise. Another algorithm proposed by Sealfon and Sotiraki in [4] also relies on synchronous communications and on the assumption that every process knows an upper bound on the system membership. Finally, in our previous paper, [28], we provided a solution for k-set agreement in asynchronous dynamic systems, with a costly assumption on the relative values of k and the system membership n.

7.2 Comparable Assumptions in the Literature

We attempt to put the strength of our assumptions into perspective by comparing them to some other existing models.

In [29], Afek and Gafni propose an implementation of read and write operations in a dynamic synchronous message passing system. Although the underlying network is assumed to be complete, in each synchronous round a subset of edges lose their messages. Therefore, such a system can be modeled as a TVG where the edges that successfully deliver their message in a round are considered active in that round. As a result, the message adversary that decides which messages will go through can be compared to a connectivity assumption. The paper defines the Traversal Path (TP) adversary as a model in which, for every synchronous round, the directed graph defined by the successfully delivered messages in this round contains a directed path passing through all the nodes. This connectivity assumption is weaker than a TVG of class 5, because traversal paths are directed paths, which implies that every process can not necessarily communicate with every other. The comparison with class 5-(α, γ) is less straightforward. On the one hand, class 5-(α, γ) implies two-way connectivity whereas a TP adversary only requires one-way connectivity. On the other hand, class $5(\alpha, \gamma)$ only requires connectivity between a limited number of nodes (as defined by the α parameter) and allows network partitioning, whereas a TP adversary connects the entire system.

In [30], Biely et al. define and implement the generalized loneliness failure detector $\mathcal{L}_k$ in a static and connected network. For this purpose, the authors use the $M^{anti(x)}$ message pattern model, which is defined by the x-Anti-Source. An x-Anti-Source is a process which is ensured to receive responses from x processes to every query it issues before it issues the next query. This definition could be used in our model: if every process in the system is an x-Anti-Source for $x \geq \lfloor \frac{n}{k+1} \rfloor + 1$, then Assumption 2 (with $\alpha = x$) and the quorum intersection property are ensured. However, the $M^{anti(x)}$ model only requires x processes to be x-Anti-Sources, which is only sufficient to implement quorums if $x = n$, since the intersection property must apply to every process in the system.

In [27], Arantes et al. present an algorithm that implements the Ω failure detector in an asynchronous TVG of class 5. To this end, the authors define the Stable Responsiveness Property ($\mathcal{SRP}$). A correct process p satisfies the $\mathcal{SRP}$ at time t if and only if, after t, all nodes in p's neighborhood receive a response from p to every one of their queries within the first α responses.

The definition of the $\mathcal{SRP}$ can be compared to the definition of an eventually winning process. Both properties enable a leader election mechanism by assuming that after some time, some process is among the first to respond to the queries of its neighbors. However, $\mathcal{SRP}$ applies to every process that shares a link with the leader after t, even for a moment, whereas the property of an eventually winning process p_l only applies to the processes of R_l, meaning those processes that interact infinitely often with p_l. Thus, in our case, a process can join the neighborhood of an eventually winning leader and leave it later on, which is not possible with a process satisfying the $\mathcal{SRP}$. But while the properties of an eventually winning leader can apply to a smaller subset of processes, those properties are stronger. Process p_l is not only required to respond to every query from its neighbors in time, it must also be the fastest to respond. Additionally, the processes within R_l are expected to communicate with each other to some extent, which is not necessary in the $\mathcal{SRP}$.

8 CONCLUSION

In this paper we adapted the existing $\Pi\Sigma_{x,y}$ failure detector to unknown dynamic systems by using the $\perp$ default value to deal with missing information and by adding connectivity properties to the failure detector definition. We obtained the $\Pi\Sigma_{\perp,x,y}$ failure detector, which is sufficient to solve k-set agreement in unknown dynamic systems with $k \geq xy$.

We then provided an algorithm implementing $\Pi\Sigma_{\perp,x,y}$ in a Time-Varying Graph of class 5-(α, γ), along with the connectivity and message pattern assumptions it relies on.

Finally, we adapted an existing algorithm to solve k-set agreement in unknown dynamic networks augmented with $\Pi\Sigma_{\perp,x,y}$ ($k \geq xy$).

Future research could attempt to further weaken the system model by removing the assumption of synchronous processes. The connectivity model would then need to be adapted, since the synchrony of processes allows the algorithm to take advantage of the time windows provided by γ-journeys. Such a change would be a challenge, because the other approaches used to ensure reachability in a TVG ([19]) rely on point to point communications, which is not applicable in an unknown network.

Another research direction would be to solve other problems in similarly weak models, such as the implementation of shared registers in a unknown dynamic message passing system.

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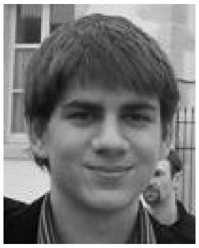

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